Math 210B Lecture 19 Notes

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1 Structure Theorem for Finitely Generated Modules over PIDs

1.1 Stripping off the torsion free part from a module

Last time, we proved the following:

Proposition 1.1. *Let R be a ring, and let $\pi : M \to F$ be a surjection of R-modules with F free. Then there exists a spitting $\iota : F \to M$ such that ι is injection and $\pi \circ \iota = \mathrm{id}_F$. Moreover, $M = \ker(\pi) \oplus \iota(F)$; i.e. F is a direct summand of M.*

Proposition 1.2. *If R is a PID and M is a finitely generated R-module, then $M \cong R^n \oplus M_{\mathrm{tor}}$ for $r = \mathrm{rank}_R(M)$.*

Proof. Let $Q = Q(R)$. Then $M \to M \otimes_R Q$ has kernel M_{tor}, so the image of $M/M_{\mathrm{tor}} \to M \otimes_R Q$ is torsion-free and hence free. So we have a surjection $M \to R^r$, where $r = \mathrm{rank}_R(M)$. Then $M/M_{\mathrm{tor}} \otimes_R Q \cong M \otimes_R Q$ with kernel M_{tor}. So $M = M_{\mathrm{tor}} \oplus R^r$. $\square$

1.2 Decomposition of the torsion part of a module

Let M be a finitely generated R-torsion module. Then $\mathrm{Ann}(M) = (x)$ for some $c \in R$ because R is a PID. The Chinese remainder theorem gives

$$R/(c) = \prod_{i=1}^r R/(\pi_i^{k_i}),$$

where $c {=} \pi_1^{k_1} \cdots \pi_r^{k_r}$ is a factorization of c into distinct irreducibles. We then get

$$M \cong M/cM \cong M \otimes_R R/(c) \cong \bigoplus_{i=1}^r M \otimes_R R/(\pi_i^{k_i}) \cong \bigoplus_{i=1}^r M/\pi_i^{k_i} M.$$

We have shown that

$$M \cong \bigoplus_{i=1}^{k_i} M_{(\pi_i)} \cong \bigoplus_{i=1}^{k} M/\pi_i^{k_i} M.$$

$R_{(\pi_i)}$ is a local ring with maximal ideal (π_i), so all of its ideals have the form (π_i^j) for $j \geq 0$ and (0). So

$$R/\pi_i^{k_i} R \cong R_{(\pi_i)}/\pi_i^{k_i} R(\pi_i)$$

has ideals (π_i^j) for $j \geq 0$ and (0).

Now let $\pi \in R$ be irreducible with $k \geq 1$, and write $\overline{R} = R/(\pi^k)$. Let M be a finitely generated $\overline{R}$-module. We split into cases. If $\overline{R} = R/(\pi)$ is a field: Then $M \cong \overline{R}^d$ for some $d \geq 0$. For the next case, we need the following.

Proposition 1.3. *If M be a finitely generated R-module with $\pi^k M = 0$, then $M \cong \bigoplus_{i=0}^{n} R/(\pi^{j_i})$ with $j_1 \geq j_2 \geq \cdots \geq j_n \geq 1$.*

We want to induct to get this, so we need the following lemma:

Lemma 1.1. *If m is a finitely generated $\overline{R}$-module and F is a maximal free $\overline{R}$-submodule, then $M = F \oplus C$ with $\pi^{k-1} C = 0$.*

Here is a case we have to watch out for:

Example 1.1. $\mathbb{Z}$ is a free $\mathbb{Z}$-module, and $2\mathbb{Z}$ is a free $\mathbb{Z}$-submodule, but the latter is not a direct summand of the former.

Lemma 1.2. *Any free $\overline{R}$-submodule of a finitely generated $\overline{R}$-module is a direct summand.*

To prove this lemma, we first have the following fact.

Proposition 1.4. *Any free $\overline{R}$-submodule of a free, finitely generated $\overline{R}$-module is a direct summand.*

Proof. Let A be a free $\overline{R}$-submodule of a finitely generated free $\overline{R}$-module B. We have the map $\iota : A \to B/\pi B$. If $a \in A$ with $\iota(a) = 0$¡ then $a \in A \cap \pi B$, so $\pi^{k-1} a = 0$. Then $a \in \pi A$. So $A/\pi A \to B/\pi B$ is an inclusion.

Then $B/\pi B = A/\pi A \oplus \overline{N}$. Last time, we showed that we can lift a basis of $B/\pi B$ containing a basis of $A/\pi A$ to a basis of B containing a basis of A. Now $B = A \oplus N$ for some N. $\square$

Assuming lemma 1 is true, we can use the fact to prove the second lemma as follows.

Proof. If $A \subseteq M$ is a free $\overline{R}$-submodule, choose F to be a maximal free submodule containing A. Then $M = F \oplus C$, and $F = A \oplus D$ by assumption, so $M = A \oplus (C \oplus D)$. $\square$

Now we can prove lemma 1.

Proof. Let $k \geq 2$. Let f be a maximal free $\overline{R}$-submodule. Let $N = M[\pi^{k-1}] = \{n \in M : \pi^{k-1}n = 0\}$. Then $\pi F \subseteq N$, and πF is a free R/π^{k-1}-submodule of N. By induction, there exists an R/π^{k-1}-submodule C such that $N = \pi F \oplus C$; here, we are using lemma 2 in the inductive step.

We claim that $M = F \oplus C$. Note that $F/\pi F \to M/N$ is an isomorphism. For injectivity, $F \cap N = \pi F$. Surjectivity follows from the maximality of F: we can lift a basis of M/N containing a basis of $F/\pi F$ to a basis of a larger or equal free $\overline{R}$-module (inside M) by the result from last time. Then $M = N + F = C + F$. Then $F \cap C = \pi F \cap C = 0$, so $M = F \oplus C$. □

1.3 The structure theorem

Theorem 1.1 (structure theorem for finitely generated modules over PIDs)**.** *Let R be a PID, and let M be a finitely generated R-module.*

1. *There exist unique $r, k \geq 0$ and nonzero proper ideals $I_1 \subseteq I_2 \subseteq \cdots I_k$. such that $M \cong R^r \oplus R/I_1 \oplus \cdots \oplus R/I_k$.*
2. *There exist unique $r, \ell \geq 0$ and distinct nonzero prime ideals p_i (up to ordering) and integers $\nu_{i,1} \geq \nu_{i,2} \geq \cdots \geq \nu_{i,m_i} \geq 1$ for some $m_i \geq 1$ such that*

$$M \cong R^r \oplus \bigoplus_{i=1}^{\ell} \bigoplus_{j=1}^{m_i} R/p_i^{\nu_{i,j}}.$$

The ideals $I_1, \dots, I_k$ are called **invariant factors**, and the $p_i^{\nu_{i,j}}$ are called **elementary divisors**.

Remark 1.1. When $R = \mathbb{Z}$, this is exactly the statement of the structure theorem for finitely generated abelian groups.

Proof. We have already proved the second part. For the first part, let $b_j = \pi_1^{\nu_{1,j}} \pi_2^{\nu_{2,j}} \cdots \pi_\ell^{\nu_{\ell,j}}$ for $j = 1, \dots, k$, where k is maximal such that $b_j \neq 1$. Here, we take $\nu_{i,j} = 0$ for $j > m_i$. Set $I_j = (b_j)$ and apply the Chinese remainder theorem:

$$R/(b_j) \cong \bigoplus_{i=1}^{\ell} R/(\pi_i^{\nu_{i,j}}).$$

Uniqueness is left as an exercise.[1] □

[1] :(